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LETTER TO THE EDITOR

**Extended integrability regime for the supersymmetric  $U$  model**

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**Abstract.** An extension of the supersymmetric  $U$  model for correlated electrons is given and integrability is established by demonstrating that the model can be constructed through the quantum inverse scattering method using an  $R$ -matrix without the difference property. Some general symmetry properties of the model are discussed and from the Bethe ansatz solution an expression for the energies is presented.

The supersymmetric (SUSY)  $U$  model was first introduced in [1] as an example of a system of correlated electrons which is integrable in one dimension as a consequence of the quantum inverse scattering method (QISM) (e.g. see [2]). Such models, which can be solved exactly by the Bethe ansatz method, are important in that the exact solutions offer non-perturbative results concerning physical behaviour. For the SUSY  $U$  model, Bethe ansatz solutions have been studied [3–9] and several analyses into the physical characteristics that the model describes have been undertaken [4, 5, 7, 10].

The construction of the SUSY  $U$  model is based on a  $R$ -matrix satisfying the Yang–Baxter equation associated with the one-parameter family of minimal typical representations of the Lie superalgebra  $gl(2|1)$ . In terms of the standard notation for electron creation, annihilation and occupation operators the local (two-site) Hamiltonian for the model reads

$$h_{i,i+1} = - \sum_{\sigma} (c_{i\sigma}^{\dagger} c_{i+1\sigma} + \text{h.c.}) (1 + U)^{1/2(n_{i,-\sigma} + n_{i+1,-\sigma})} + U (c_{i\downarrow}^{\dagger} c_{i\uparrow}^{\dagger} c_{i+1\uparrow} c_{i+1\downarrow} + \text{h.c.}) + U (n_{i\uparrow} n_{i\downarrow} + n_{i+1\uparrow} n_{i+1\downarrow}) - 2 + n_{i\uparrow} + n_{i\downarrow} + n_{i+1\uparrow} + n_{i+1\downarrow}$$

where  $U$  is an arbitrary free parameter. The local Hamiltonian is also invariant with respect to the Lie superalgebra  $gl(2|1)$  (hence the name). Below, an extension of this model will be derived in such a way that integrability is maintained. The local Hamiltonian of the new model reads

$$h_{i,i+1} = - \sum_{\sigma} ((1 - it) c_{i\sigma}^{\dagger} c_{i+1\sigma} + \text{h.c.}) (1 + U)^{1/2(n_{i,-\sigma} + n_{i+1,-\sigma})} + U ((1 - it) c_{i\downarrow}^{\dagger} c_{i\uparrow}^{\dagger} c_{i+1\uparrow} c_{i+1\downarrow} + \text{h.c.}) + U (n_{i\uparrow} n_{i\downarrow} + n_{i+1\uparrow} n_{i+1\downarrow}) - 2 + n_{i\uparrow} + n_{i\downarrow} + n_{i+1\uparrow} + n_{i+1\downarrow} \tag{1}$$

where now  $t$  is an additional free variable which when chosen to be real (along with  $U$  real and  $U > -1$ ) results in a Hermitian Hamiltonian. For the case  $t = 0$  the usual SUSY  $U$  model

is recovered. The extended model bears some similarity with the multiparametric SUSY  $U$  model constructed in [11] but is in fact inherently different.

The construction of the above model is through the use of a solution of the Yang–Baxter equation without difference property in the spectral parameter. It is known that the Hubbard model may be derived via the QISM using an  $R$ -matrix which is also without the difference property [12, 13]. However, for the Hubbard model the Lax operator is given as a particular coupling of two auxiliary Lax operators of six-vertex type. The construction employed here appears more akin to the generalized chiral Potts models given in [14] based on representations of quantum algebras at roots of unity. For these models and the one discussed here the spectral parameters without difference property originate from the *representation* of the underlying algebraic structure.

In order to demonstrate integrability of this model, we begin with the rational limit of the  $U_q(gl(2|1))$  invariant (i.e.  $gl(2|1)$  invariant) solution of the Yang–Baxter equation with additional spectral parameters constructed in [15, 16]. This solution may be written in the form

$$R(u, \beta, \alpha) = \frac{u - \alpha - \beta}{u + \alpha + \beta} P_1 + P_2 + \frac{u + \alpha + \beta + 2}{u - \alpha - \beta - 2} P_3 \quad (2)$$

and satisfies the Yang–Baxter equation

$$R_{12}(u - v, \beta, \gamma) R_{13}(u, \beta, \alpha) R_{23}(v, \gamma, \alpha) = R_{23}(v, \gamma, \alpha) R_{13}(u, \beta, \alpha) R_{12}(u - v, \beta, \gamma).$$

Note that this solution of the Yang–Baxter equation has multiple spectral parameters, one of which has the difference property while the other two do not. It is worth remarking at this point that the above solution is just one of a plethora of solutions of this type which arise naturally as a consequence of the representation theory of the type-I quantum superalgebras [16, 17].

In the above expression for the  $R$ -matrix the operators  $P_i$  are  $gl(2|1)$  invariant projection operators. To explain their actions we begin by recalling that  $gl(2|1)$  has generators  $E_j^i$ ,  $i, j = 1, 2, 3$  satisfying the super commutator relations

$$[E_j^i, E_l^k] = \delta_j^k E_l^i - (-1)^{((i+1+j)(k+l))} \delta_l^i E_j^k.$$

Above,  $[1] = [2] = 0, [3] = 1$ . Choosing a four-dimensional space with basis  $\{|i\rangle : i = 1, 2, 3, 4\}$ , a representation  $\pi_\alpha$  of  $gl(2|1)$  acting on this space exists with the action of the generators given by

$$\begin{aligned} E_2^1 &= |2\rangle\langle 3| & E_1^2 &= |3\rangle\langle 2| & E_1^1 &= -|3\rangle\langle 3| - |4\rangle\langle 4| & E_2^2 &= -|2\rangle\langle 2| - |4\rangle\langle 4| \\ E_3^2 &= \sqrt{\alpha}|1\rangle\langle 2| + \sqrt{\alpha+1}|3\rangle\langle 4| & E_2^3 &= \sqrt{\alpha}|2\rangle\langle 1| + \sqrt{\alpha+1}|4\rangle\langle 3| \\ E_3^1 &= -\sqrt{\alpha}|1\rangle\langle 3| + \sqrt{\alpha+1}|2\rangle\langle 4| & E_1^3 &= -\sqrt{\alpha}|3\rangle\langle 1| + \sqrt{\alpha+1}|4\rangle\langle 2| \\ E_3^3 &= \alpha|1\rangle\langle 1| + (\alpha+1)|2\rangle\langle 2| + |3\rangle\langle 3| + (\alpha+2)|4\rangle\langle 4|. \end{aligned}$$

Above, the states  $|1\rangle, |4\rangle$  are bosonic and  $|2\rangle, |3\rangle$  are fermionic. The highest weight state is  $|1\rangle$  with weight  $(0, 0|\alpha)$ . It is significant here that  $\alpha$  is a free complex parameter. Considering the tensor product representation  $\pi_\beta \otimes \pi_\alpha$  for generic values of  $\alpha$  and  $\beta$ , then  $P_1$  projects onto the irreducible submodule with (unnormalized) basis vectors

$$\begin{aligned} |\Psi_1^1\rangle &= |1\rangle \otimes |1\rangle & |\Psi_2^1\rangle &= \sqrt{\beta}|2\rangle \otimes |1\rangle + \sqrt{\alpha}|1\rangle \otimes |2\rangle \\ |\Psi_3^1\rangle &= \sqrt{\beta}|3\rangle \otimes |1\rangle + \sqrt{\alpha}|1\rangle \otimes |3\rangle \\ |\Psi_4^1\rangle &= \sqrt{\beta(\beta+1)}|4\rangle \otimes |1\rangle + \sqrt{\alpha(\alpha+1)}|1\rangle \otimes |4\rangle + \sqrt{\alpha\beta}(|2\rangle \otimes |3\rangle - |3\rangle \otimes |2\rangle) \end{aligned} \quad (3)$$

while  $P_3$  projects onto the irreducible space spanned by

$$\begin{aligned} |\Psi_1^2\rangle &= \sqrt{\alpha(\alpha+1)}|4\rangle \otimes |1\rangle + \sqrt{\beta(\beta+1)}|1\rangle \otimes |4\rangle \\ &+ \sqrt{(\alpha+1)(\beta+1)}(-|2\rangle \otimes |3\rangle + |3\rangle \otimes |2\rangle) \end{aligned}$$

$$\begin{aligned}
 |\Psi_2^2\rangle &= \sqrt{\beta+1}|2\rangle \otimes |4\rangle + \sqrt{\alpha+1}|4\rangle \otimes |2\rangle) \\
 |\Psi_3^2\rangle &= \sqrt{\beta+1}|3\rangle \otimes |4\rangle + \sqrt{\alpha+1}|4\rangle \otimes |3\rangle) \quad |\Psi_4^2\rangle = |4\rangle \otimes |4\rangle.
 \end{aligned}$$

The projector  $P_2$  is obtained from  $P_2 = I - P_1 - P_3$ . From this solution of the Yang–Baxter equation we may now construct the transfer matrix

$$t(u, \beta, \alpha) = \text{str}_0 R_{0L}(u, \beta, \alpha) \dots R_{02}(u, \beta, \alpha) R_{01}(u, \beta, \alpha)$$

which forms a commuting family in two variables; namely,

$$[t(u, \beta, \alpha), t(v, \gamma, \alpha)] = 0$$

and thus  $t(u, \beta, \alpha)$  can be diagonalized independently of both  $u$  and  $\beta$ . In fact, the diagonalization of this transfer matrix has already been treated in [8] with the result that the eigenvalues are given by

$$\Lambda(u, \beta, \alpha) = \left(\frac{u - \alpha - \beta}{u + \alpha + \beta}\right)^L \bar{\Lambda}(u, \beta, \alpha) \tag{4}$$

with

$$\begin{aligned}
 \bar{\Lambda}(u, \beta, \alpha) &= \prod_{i=1}^n \frac{u - \lambda_i + \beta}{u - \lambda_i - \beta} + \left(\frac{u + \alpha - \beta}{u - \alpha - \beta} \cdot \frac{u + \alpha - \beta - 2}{u - \alpha - \beta - 2}\right)^L \prod_{i=1}^n \frac{u - \lambda_i + \beta}{u - \lambda_i - \beta - 2} \\
 &\quad - \left(\frac{u + \alpha - \beta}{u - \alpha - \beta}\right)^L \left\{ \prod_{i=1}^n \frac{u - \lambda_i + \beta}{u - \lambda_i - \beta} \prod_{j=1}^m \frac{u - v_j - \beta + 1}{u - v_j - \beta - 1} \right. \\
 &\quad \left. + \prod_{i=1}^n \frac{u - \lambda_i + \beta}{u - \lambda_i - \beta - 2} \prod_{j=1}^m \frac{u - v_j - \beta - 3}{u - v_j - \beta - 1} \right\} \tag{5}
 \end{aligned}$$

such that the parameters  $\lambda_i, v_j$  are solutions of the Bethe ansatz equations

$$\begin{aligned}
 \left(\frac{\lambda_k + \alpha}{\lambda_k - \alpha}\right)^L &= \prod_{j=1}^m \frac{\lambda_k - v_j - 1}{\lambda_k - v_j + 1} \quad k = 1, \dots, n \\
 \prod_{k=1}^n \frac{\lambda_k - v_i + 1}{\lambda_k - v_i - 1} &= - \prod_{j=1}^m \frac{v_j - v_i + 2}{v_j - v_i - 2} \quad i = 1, \dots, m.
 \end{aligned}$$

The  $R$ -matrix possesses the property  $R(0, \alpha, \alpha) = -P$  where  $P$  is the  $\mathbb{Z}_2$ -graded permutation operator. From here on in we make the parametrization

$$\beta = it u + \alpha$$

and fix  $t$  and  $\alpha$ . Writing the  $R$ -matrix now as a function of only  $u$  we have that

$$R(u = 0) = -P$$

and thus by the standard approach of the QISM [2] we can construct a closed periodic quantum model where the Hamiltonian is the logarithmic derivative of the transfer matrix and expressible as

$$H = \sum_{i=1}^{L-1} h_{i(i+1)} + h_{L1}$$

with the two-site Hamiltonian given by

$$h = -P \cdot \left. \frac{d}{du} R(u) \right|_{u=0}.$$

In terms of the projections operators (cf (2)) we have

$$h = \frac{-1}{\alpha} P_1 + \frac{1}{\alpha + 1} P_3 - 2it P \frac{dP_2}{d\beta} \Big|_{\beta=\alpha}$$

which when expressed in terms of Fermi operators leads to (1) with  $U = \alpha^{-1}$  and an overall normalization factor of  $-2(\alpha + 1)$  included for convenience. The energies of the Bethe states are obtained from the transfer matrix eigenvalues

$$E = -2(\alpha + 1) \Lambda^{-1} \frac{d\Lambda}{du} \Big|_{u=0} = \frac{2L(\alpha + 1)}{\alpha} + 4(\alpha + 1) \sum_{i=1}^n \frac{\alpha + it\lambda_i}{\lambda_i^2 - \alpha^2}.$$

In conclusion we discuss some remarkable properties this model possesses. By construction, the transfer matrix from which it is derived is  $gl(2|1)$  invariant. In fact, one may show that the eigenstates obtained by the algebraic Bethe ansatz method are highest weight states in complete analogy with other  $gl(2|1)$  invariant models studied in [18–20]. However, the local Hamiltonians are *not*  $gl(2|1)$  invariant. Only for the global system is the supersymmetry present. Regardless, it is still possible to add arbitrary chemical potential and magnetic field terms to the local Hamiltonian (1) which do not violate the integrability.

Construction of the model for  $L = 2$  yields the usual supersymmetric  $U$  model, as does the construction on an open chain using (an appropriate modification of) Sklyanin's approach [21] where the boundary  $K$ -matrices are chosen to be trivial. These unusual scenarios are a consequence of the property

$$P \frac{dP_2}{d\beta} \Big|_{\beta=\alpha} = - \frac{dP_2}{d\beta} \Big|_{\beta=\alpha} P$$

which is precisely the symmetry breaking term for the local Hamiltonians, and the presence of which also implies that

$$h_{i(i+1)} \neq h_{(i+1)i}.$$

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